SHORT HARMONIC SUPERFIELDS AND LIGHT-CONE GAUGE IN SUPER-YANG-MILLS EQUATIONS [1]

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Abstract

We analyze the superfield equations of the 4-dimensional N=2 and N=4 SYM-theories using light-cone gauge conditions and the harmonic-superspace approach. The harmonic superfield equations of motion are drastically simplified in this gauge, in particular, the basic harmonic-superfield matrices and the corresponding harmonic analytic gauge connections become nilpotent on-shell.

1 Introduction

The $D = 4,\ N = 2$ harmonic-superspace (HSS) has been introduced first for the solution of the off-shell superfield constraints [1]. The superfield action and equations of motion of the $N = 2$ super-Yang-Mills (SYM) theory have been also constructed in this superspace [2, 3]. In the standard harmonic formulation of this theory, the basic harmonic connection satisfies the conditions of the Grassmann (G-) analyticity, and the 2-nd one (via the zero-curvature condition) appears to be a nonlinear function of the basic connection. The $N = 2$ equation of motion is linearly dependent on the 2-nd harmonic connection, but it is the nonlinear equation for the basic connection. It has been shown in Ref.[4] that one can alternatively choose the 2-nd harmonic connection as a basic superfield, so that the dynamical G-analyticity condition (or the Grassmann-harmonic zero-curvature condition) for the first connection becomes a new equation of motion. In the harmonic approach to the $D = 4,\ N = 3$ SYM-theory [5], the $SU(3)/U(1) \times U(1)$ harmonics have been used for the covariant reduction of the spinor coordinates and derivatives and for the off-shell description of the SYM-theory in terms of the corresponding G-analytic superfields. Moreover, it was shown that the $N = 3$ SYM-constraints in the ordinary superspace [6] can been transformed to the zero-curvature equations for the analytic harmonic gauge connections (see, also the alternative formalism [7]). Harmonic superspaces of the $D = 4, N = 4$ supersymmetry have been considered in Refs. [8, 9, 10].

The short on-shell harmonic superfields describing the Abelian $N = 2, 3$ and 4 SYM-multiplets satisfy the constraints of chirality or different types of harmonic and Grassmann analyticities. It will be shown that the short harmonic superfields and the nonlinear harmonic equations can be used to describe the classical solutions of the non-Abelian SYM-equations for different N.

We shall analyze the classical solutions of the harmonic-superfield equations using the convenient light-cone gauge conditions for superfield connections (see, e.g. [11, 12, 13]). As it has been shown in Refs. [14] these gauge conditions yield the nilpotent superfield

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matrices in the bridge representation of the $N=3$ SYM-theory and simplify drastically the harmonic-superfield equations. We shall consider analogous nilpotent gauge conditions for the $N=2$ and $N=4$ theories and the corresponding simplifications of the harmonic SYM-equations.

2 Solving $D=4,\ N=2$ SYM equations in harmonic superspace

The covariant coordinates of the $D=4,\ N=2$ superspace are

$$z^M = (x^{\alpha\dot{\alpha}}, \theta^\alpha_i, \bar{\theta}^{i\dot{\alpha}}) \ , \tag{2.1}$$

where $\alpha,\ \dot{\alpha}$ are the $SL(2,C)$ indices and $i=1,2$ are indices of the fundamental representations of the group $SU(2)$.

We shall study solutions of the SYM-equations using the non-covariant notation

$$\begin{aligned} &x^{\ddagger} \equiv x^{1\dot{1}} = t + x^3 \ , \quad x^{=} \equiv x^{2\dot{2}} = t - x^3 \ , \quad y \equiv x^{1\dot{2}} = x^1 + ix^2 \ , \\ &\bar{y} \equiv x^{2\dot{1}} = x^1 - ix^2 \ , \quad (\theta^+_i,\ \theta^-_i) \equiv \theta^\alpha_i \ , \quad (\bar{\theta}^{i+},\ \bar{\theta}^{i-}) \equiv \bar{\theta}^{i\dot{\alpha}} \ . \end{aligned} \tag{2.2}$$

suitable when the Lorenz symmetry is reduced to $SO(1,1)$. The general $N=2$ superspace has the odd dimension (4,4) in this notation.

Let us define the gauge connections $A(z)$ and the corresponding covariant derivatives ∇ in the $(4|4,4)$-dimensional superspace, then the $D=4,\ N=2$ SYM-constraints [15] have the following reduced-symmetry form:

$$\{\nabla^k_+, \nabla^l_+\} = 0 \ , \quad \{\bar{\nabla}_{k+}, \bar{\nabla}_{l+}\} = 0 \ , \quad \{\nabla^k_+, \bar{\nabla}_{l+}\} = 2i\delta^k_l \nabla_{\ddagger} \ , \tag{2.3}$$
$$\{\nabla^k_+, \nabla^l_-\} = \varepsilon^{kl}\bar{W} \ , \quad \{\nabla^k_+, \bar{\nabla}_{l-}\} = 2i\delta^k_l \nabla_y \ , \tag{2.4}$$
$$\{\nabla^k_-, \bar{\nabla}_{l+}\} = 2i\delta^k_l \bar{\nabla}_y \ , \quad \{\bar{\nabla}_{k+}, \bar{\nabla}_{l-}\} = \varepsilon_{kl} W \ , \tag{2.5}$$
$$\{\nabla^k_-, \nabla^l_-\} = 0 \ , \quad \{\bar{\nabla}_{k-}, \bar{\nabla}_{l-}\} = 0 \ , \quad \{\nabla^k_-, \bar{\nabla}_{l-}\} = 2i\delta^k_l \nabla_{=} \ , \tag{2.6}$$

where W and $\bar{W}$ are the gauge-covariant superfield strengthes constructed from the gauge connections. In particular, this reduced form of the 4D constraints is convenient for the study of dimensional reduction.

The Bianchi identities yield the following relations for the superfield strengthes

$$\bar{\nabla}_{i\pm} W = 0 \ , \quad \nabla^i_\pm \bar{W} = 0 \ , \tag{2.7}$$
$$\nabla^{(i}_+ \nabla^{k)}_- W = \bar{\nabla}^{(i}_- \bar{\nabla}^{k)}_+ \bar{W} \ . \tag{2.8}$$

Let us analyze first Eqs.(2.3) together with the relations

$$[\nabla^k_+, \nabla_{\ddagger}] = [\bar{\nabla}_{k+}, \nabla_{\ddagger}] = 0 \ . \tag{2.9}$$

These equations for the positive-helicity connections have the pure gauge solutions only, so the simplest light-cone gauge condition can be taken in the following form:

$$A^k_+ = 0 \ , \quad \bar{A}_{k+} = 0 \ , \quad A_{\ddagger} = 0 \ . \tag{2.10}$$

Let us consider the covariantly chiral superfield strength in the gauge (2.10)

$$W = \frac{1}{2}\bar{D}^k_+ \bar{A}_{k-} \ . \tag{2.11}$$

The nonlinear superfield equation of motion for the $N = 2$ non-Abelian SYM-theory has dimension $l = -2$

$$D^{(i}_+ \nabla^{k)}_- W = 0 \ . \tag{2.12}$$

In distinction with the case $N = 3$ [6], this dynamical equation is independent of constraints (2.3-2.6).

The Lorenz-covariant $SU(2)/U(1)$ harmonic superspace was introduced in Ref.[1] for the off-shell description of the $4D$ $N = 2$ SYM-theory, supergravity and hypermultiplets.

Now we shall study this harmonic superspace in another (reduced-symmetry) representation which allows us to consider the non-covariant gauges and the dimensional reduction.

The $SU(2)/U(1)$ harmonics [1] parametrize the sphere S^2. They form an $SU(2)$ matrix u^I_i and are defined modulo $U(1)$

$$u^1_i = u_{2i} \equiv u^+_i \ , \quad u^2_i = -u_{1i} \equiv u^-_i \ . \tag{2.13}$$

where $i = 1, 2$ is the index of the fundamental representation of $SU(2)$. Note that we use the non-standard notation of the $U(1)$-charges in comparison to Ref.[1] since indices $\pm$ are reserved for the $SO(1, 1)$ weights in this paper.

The $SU(2)$-invariant harmonic derivatives act on the harmonics $\partial^I_J u^K_i = \delta^K_J u^I_i$.

We can define a non-covariant notation for coordinates in the analytic harmonic superspace $H(4, 2|2, 2)$

$$\begin{aligned}
&\zeta = (X^{\ddagger},\ X^{=},\ Y,\ \bar{Y}\ |\theta^{\pm}_2\ ,\bar{\theta}^{1\pm})\ , \qquad X^{\ddagger} = x^{\ddagger} + i(\theta^+_2\bar{\theta}^{2+} - \theta^+_1\bar{\theta}^{1+})\ ,\\
&X^{=} = x^{=} + i(\theta^-_2\bar{\theta}^{2-} - \theta^-_1\bar{\theta}^{1-})\ , \quad Y = y + i(\theta^+_2\bar{\theta}^{2-} - \theta^+_1\bar{\theta}^{1-})\ ,\\
&\bar{Y} = \bar{y} + i(\theta^-_2\bar{\theta}^{2+} - \theta^-_1\bar{\theta}^{1+})\ , \quad \theta^{\pm}_I = \theta^{\pm}_k u^k_I\ , \quad \bar{\theta}^{I\pm} = \bar{\theta}^{\pm k} u^I_k\ .
\end{aligned} \tag{2.14}$$

The special $SU(2)$-covariant conjugation of harmonics preserves the $U(1)$-charges

$$\widetilde{u^1_i} = u^i_2\ , \quad \widetilde{u^2_i} = -u^i_1\ , \quad \widetilde{u^i_2} = -u^1_i\ , \quad \widetilde{u^i_1} = u^2_i\ . \tag{2.15}$$

On the harmonic derivatives of an arbitrary harmonic function $f(u)$ this conjugation acts as follows

$$\widetilde{\partial^1_2 f} = \partial^1_2 \widetilde{f}\ , \quad \widetilde{\partial^2_1 f} = \partial^2_1 \widetilde{f}\ . \tag{2.16}$$

The tilde-conjugation of the odd analytic coordinates has the following form:

$$\theta^{\pm}_1 \ \rightarrow\ \bar{\theta}^{2\pm}\ , \quad \bar{\theta}^{2\pm} \ \rightarrow\ -\,\theta^{\pm}_1\ , \quad \theta^{\pm}_2 \ \rightarrow\ -\,\bar{\theta}^{1\pm}\ , \quad \bar{\theta}^{1\pm} \ \rightarrow\ \theta^{\pm}_2\ . \tag{2.17}$$

Coordinates $X^{\ddagger}$ and $X^{=}$ are real and $\widetilde{Y} = \bar{Y}$.

The corresponding CR-structure involves the derivatives

$$D^1_{\pm},\ \bar{D}_{2\pm},\ D^1_2 \tag{2.18}$$

which have the following explicit form in these coordinates:

$$D^1_\pm = \partial^1_\pm \ , \quad \bar{D}_{2\pm} = \bar{\partial}_{2\pm} \ , \tag{2.19}$$

$$D^1_2 = \partial^1_2 + 2i\theta^+_2\bar{\theta}^{1+}\partial_{\ddagger} + 2i\theta^+_2\bar{\theta}^{1-}\partial_Y + 2i\theta^-_2\bar{\theta}^{1+}\bar{\partial}_Y + 2i\theta^-_2\bar{\theta}^{1-}\partial_{=}$$
$$-\theta^+_2\partial^1_+ - \theta^-_2\partial^1_- + \bar{\theta}^{1+}\bar{\partial}_{2+} + \bar{\theta}^{1-}\bar{\partial}_{2-} \ , \tag{2.20}$$

where $\partial_{\ddagger} = \partial/\partial X^{\ddagger}$, $\partial_{=} = \partial/\partial X^{=}$, $\partial_Y = \partial/\partial Y$ and $\bar{\partial}_Y = \partial/\partial\bar{Y}$.

It is crucial that we start from the light-cone gauge conditions for the $N=2$ SYM-connections which break $SL(2,C)$, but preserve $SU(2)$. Consider the harmonic transform of the covariant Grassmann derivatives via the projections on the $SU(2)$-harmonics. As result we get so called harmonized Grassmann covariant derivatives

$$\nabla^I_+ \equiv u^I_i D^i_+ = D^I_+ \ , \quad \bar{\nabla}_{I+} \equiv u^i_I \bar{D}_{i+} = \bar{D}_{I+} \ , \quad \{D^I_+, \bar{D}_{K+}\} = 2i\delta^I_K\partial_{\ddagger} \ , \tag{2.21}$$

$$\nabla^I_- \equiv u^I_i \nabla^i_- = D^I_- + \mathcal{A}^I_- \ , \quad \bar{\nabla}_{I-} \equiv u^i_I \nabla_{i-} = \bar{D}_{I-} + \bar{\mathcal{A}}_{I-} \ . \tag{2.22}$$

The $SU(2)$-harmonic projections of the superfield constraints (2.4-2.6) can be derived from the basic set of the $N=2$ zero-curvature (or G-integrability) conditions for two harmonized Grassmann connections:

$$D^1_+\mathcal{A}^1_- = \bar{D}_{2+}\mathcal{A}^1_- = D^1_+\bar{\mathcal{A}}_{2-} = \bar{D}_{2+}\bar{\mathcal{A}}_{2-} = 0 \ , \tag{2.23}$$

$$D^1_-\mathcal{A}^1_- + (\mathcal{A}^1_-)^2 = 0 \ , \quad \bar{D}_{2-}\bar{\mathcal{A}}_{2-} + (\bar{\mathcal{A}}_{2-})^2 = 0 \ ,$$
$$D^1_-\bar{\mathcal{A}}_{2-} + \bar{D}_{2-}\mathcal{A}^1_- + \{\mathcal{A}^1_-, \bar{\mathcal{A}}_{2-}\} = 0 \ . \tag{2.24}$$

All projections of the SYM-constraints can be obtained acting by the harmonic derivatives D^2_1 on this basic set of conditions.

The G-integrability equations (2.24) have a very simple general solution, namely

$$\mathcal{A}^1_-(v) = e^{-v}D^1_- e^v \ , \quad \bar{\mathcal{A}}_{2-}(v) = e^{-v}\bar{D}_{2-}e^v \ , \tag{2.25}$$

where *the bridge* v is a superfield matrix satisfying the additional constraint

$$(D^1_+, \ \bar{D}_{2+})v = 0 \ , \tag{2.26}$$

which is compatible with the light-cone representation (2.21). Thus, v does not depend on θ^+_1 and $\bar{\theta}^{2+}$ in analytic coordinates (2.14).

Consider the gauge transformations of bridge v

$$e^v \ \Rightarrow \ e^\lambda e^v e^\tau \ , \tag{2.27}$$

where $\lambda \in H(4,6|2,2)$ is a G-analytic matrix parameter, and constrained parameter τ does not depend on harmonics. Matrix e^v can be interpreted as a map of gauge superfields $A^k_\pm, \bar{A}_{k\pm}$ defined in the central basis (CB) to those in the analytic basis (AB).

The SYM-constraints for v due to Eqs. (2.25) are reduced to the following harmonic differential conditions for the basic Grassmann connections:

$$D^1_2\left(\mathcal{A}^1_-(v), \ \bar{\mathcal{A}}_{2-}(v)\right) = 0 \ . \tag{2.28}$$

Note that these *H-analyticity* relations are trivial for Grassmann connections in the CB-superfield representation, but they become the nontrivial differential equations for bridge v. Equations (2.28) are completely equivalent to the G-integrabity equations (2.24), and they can be treated as a new representation of the SYM-constraints.

It is not difficult to built all Grassmann CB-connections in terms of basic ones

$$\begin{aligned} &\mathcal{A}^2_-(v) = D^2_1 \mathcal{A}^1_-(v) \ , \\ &\bar{\mathcal{A}}_{1-}(v) = -D^2_1 \bar{\mathcal{A}}_{2-}(v) \ . \end{aligned} \tag{2.29}$$

The non-Abelian CB-superfield strengths are

$$\begin{aligned} &W = \bar{D}_{1+} \bar{\mathcal{A}}_{2-}(v) = -\bar{D}_{2+} \bar{\mathcal{A}}_{1-}(v) \ , \\ &\bar{W} = -D^2_+ \mathcal{A}^1_-(v) = D^1_+ \mathcal{A}^2_-(v) \ . \end{aligned} \tag{2.30}$$

Now we shall determine the explicit form of bridge v. Using the off-shell $(4,4)$-analytic λ-transformations $\delta v = \lambda + \frac{1}{2}[\lambda, v] + \ldots$ one can choose the following non-supersymmetric nilpotent gauge condition for v

$$v = \theta^-_1 b^1 + \bar{\theta}^{2-} \bar{b}_2 + \theta^-_1 \bar{\theta}^{2-} d^1_2 \ , \quad v^2 = \theta^-_1 \bar{\theta}^{2-} [\bar{b}_2, b^1] \ , \quad v^3 = 0 \ , \tag{2.31}$$

$$e^{-v} = I - v + \frac{1}{2} v^2 = I - \theta^-_1 b^1 - \bar{\theta}^{2-} \bar{b}_2 + \theta^-_1 \bar{\theta}^{2-} (\frac{1}{2}[\bar{b}_2, b^1] - d^1_2) \ , \tag{2.32}$$

where fermionic matrices $b^1, \bar{b}_2$ and bosonic matrix d^1_2 are analytic functions of coordinates ζ (2.14).

Note that the analogous nilpotent gauge for the $N = 3$ harmonic bridge has been introduced in Ref.[14]. This gauge for v combines the actions of both the analytic gauge group and the CB-gauge group.

We use the harmonic tilde-conjugation (2.15, 2.17) to describe the reality conditions for the gauge superfields in HSS. For instance, the Hermitian conjugation $\dagger$ of the superfield matrices includes transposition and this conjugation. The conditions for bridge v in the gauge group $SU(n)$ are $v^\dagger = -v$ and $\mathrm{Tr}\ v = 0$, so that matrices $b^1, \bar{b}_2$ and d^1_2 have the following properties in the group $SU(n)$:

$$\mathrm{Tr}\ b^1 = 0 \ , \quad \mathrm{Tr}\ \bar{b}_2 = 0 \ , \quad \mathrm{Tr}\ d^1_2 = 0 \ , \tag{2.33}$$

$$(b^1)^\dagger = \bar{b}_2 \ , \quad (\bar{b}_2)^\dagger = -b^1 \ , \quad (d^1_2)^\dagger = d^1_2 \ . \tag{2.34}$$

Note that the last property is connected with relation $(\theta^-_1 \bar{\theta}^{2-})^\dagger = -\theta^-_1 \bar{\theta}^{2-}$.

It is useful to consider the explicit parametrization of the Grassmann connection $\mathcal{A}^1_-(v)$ and $\bar{\mathcal{A}}_{3-}(v)$ in terms of basic analytic matrices (2.31)

$$\mathcal{A}^1_-(v) \equiv e^{-v} D^1_- e^v = b^1 - \theta^-_1 (b^1)^2 + \bar{\theta}^{2-} f^1_2 + \theta^-_1 \bar{\theta}^{2-} [b^1, f^1_2] \, , \tag{2.35}$$

$$\bar{\mathcal{A}}_{2-} \equiv e^{-v} \bar{D}_{2-} e^v = \bar{b}_2 + \theta^-_1 \bar{f}^1_2 - \bar{\theta}^{2-} (\bar{b}_2)^2 + \theta^-_1 \bar{\theta}^{2-} [\bar{f}^1_2, \bar{b}_2] \ , \tag{2.36}$$

where the following auxiliary superfields are introduced:

$$f^1_2 = d^1_2 - \frac{1}{2}\{b^1, \bar{b}_2\} \ , \quad \bar{f}^1_2 = d^1_2 + \frac{1}{2}\{b^1, \bar{b}_2\} \ . \tag{2.37}$$

Equation $D^1_2 \mathcal{A}^1_-(v) = 0$ generates the following independent relations for the (2,2)-analytic matrices:

$$D^1_2 b^1 = -\theta^-_2 (b^1)^2 - \bar{\theta}^{1-} f^1_2 \ , \tag{2.38}$$
$$D^1_2 f^1_2 = \theta^-_2 [f^1_2, b^1] \ . \tag{2.39}$$

Equation $D^1_2 \bar{\mathcal{A}}_{2-}(v) = 0$ gives the conjugated relations.

It is easily to show that these equations yield the subsidiary conditions for the coefficient functions

$$D^1_2 D^1_2 b^1 = 2\theta^-_2 \bar{\theta}^{1-} [f^1_2, b^1] \ , \quad (D^1_2)^3 b^1 = 0 \ , \quad (D^1_2)^2 f^1_2 = 0 \ . \tag{2.40}$$

In order to understand more deeply the geometric structure of our harmonic-superspace solutions it is useful to represent them in the analytic basis. Remember that the following covariant Grassmann derivatives are flat in the analytic representation of the gauge group before the gauge fixing:

$$e^v \nabla^1_\pm e^{-v} = D^1_\pm \ , \quad e^v \bar{\nabla}_{2\pm} e^{-v} = \bar{D}_{2\pm} \ . \tag{2.41}$$

The harmonic transform of the covariant derivatives via matrix e^v (2.25) determines in AB the on-shell harmonic connections as a function of v

$$\nabla^I_K \equiv e^v D^I_K e^{-v} = D^I_K + V^I_K(v) \ , \quad I \neq K \ . \tag{2.42}$$

In the off-shell $N = 2$ formalism [1], the connection V^1_2 is the basic G-analytic prepotential by construction. The nonlinear solution for the 2-nd harmonic connection $V^2_1(V^1_2)$ can be found explicitly [3]. The Grassmann connections can be constructed via this connection

$$a^2_\pm = -D^1_\pm V^2_1 \ , \quad \bar{a}_{1\pm} = \bar{D}_{2\pm} V^2_1 \ . \tag{2.43}$$

In this analytic representation, the equation of motion of the SYM-theory has the following form:

$$D^1_+ D^1_- \bar{D}_{2+} \bar{D}_{2-} V^2_1(V^1_2) = 0 \ . \tag{2.44}$$

Stress that this dynamical equation is nonlinear, while the Grassmann analyticity of V^1_2 is pure kinematic (off-shell).

The alternative non-analytic representation of harmonic-superspace equations for the $N = 2$ SYM-theory has been proposed in Ref.[4]. The basic superfield variable of this dual representation is non-analytic connection V^2_1, then equation (2.44) becomes the simple linear constraint. The basic dynamical equation of this representation has the form of the Grassmann-harmonic zero-curvature equation, so the structure of the $N = 2$ SYM-equation become similar to the structure of the $= 3$ equation in the HSS-approach.

Thus, the bridge matrix can be interpreted as the 3-rd possible form of the basic superfield variable which admits the most simple gauge conditions (2.31).

The analytic SYM-equations (2.38,2.39) are equivalent to the following relation :

$$V^1_2 \equiv e^v D^1_2 e^{-v} = \theta^-_2 b^1 - \bar{\theta}^{1-} \bar{b}_2 \ , \tag{2.45}$$

which determines the nilpotent analytic connection in our representation. One can check straightforwardly the properties of this representation

$$(V_2^1)^3 = 0 \ , \quad D_2^1 V_2^1 = \theta_2^- \bar{\theta}^{1-} \{b^1, \bar{b}_2\} \ , \quad (D_2^1)^2 v_2^1 = 0 \ . \tag{2.46}$$

The bridge representation of the harmonic connection V_1^2 is

$$V_1^2(v) = -\theta_1^- D_1^2 b^1 - \bar{\theta}^{2-} D_1^2 \bar{b}_2 + \theta_1^- \bar{\theta}^{2-} \left(-D_1^2 d_2^1 + \frac{1}{2}\{b^1, D_1^2 \bar{b}_2\} - \frac{1}{2}\{D_1^2 b^1, \bar{b}_2\} \right) \ . \tag{2.47}$$

Equation (2.12) transforms to the following simple analytic equation in the bridge representation:

$$\begin{aligned} &\bar{D}_{2+}\bar{D}_{2-} D_+^1 D_-^1 V_1^2(v) = 2i \bar{D}_{2-} D_-^1 \left(e^v \partial_{\ddagger} e^{-v} \right) \\ &= 2i \left(-\partial_{\ddagger} d_2^1 + \frac{1}{2}\{b^1, \partial_{\ddagger} \bar{b}_2\} - \frac{1}{2}\{\partial_{\ddagger} b^1, \bar{b}_2\} \right) = 0 \ , \end{aligned} \tag{2.48}$$

where $\partial_{\ddagger} = \partial/\partial X^{\ddagger}$. This equation can be interpreted as a solvable linear equation for superfield d_2^1, and one should analyze it together with the harmonic equations for the basic analytic matrices.

Note that the simple Ansatz $\partial_{\ddagger} d_2^1 = \partial_{\ddagger} b^1 = \partial_{\ddagger} \bar{b}_2 = 0$ yields the partial solution of this equation. The corresponding solutions of the $N = 2$ SYM-equations can be obtained from the pure harmonic equations (2.38,2.39).

It is important that these differential harmonic equations contain the nilpotent elements θ_2^- or $\bar{\theta}^{1-}$ in the nonlinear parts, so the simplest iteration procedure for finding their solutions can be obtained via a partial Grassmann decomposition. This decomposition generates the finite set of solvable linear iterative equations.

3 Harmonic-superspace formulation of $N = 4$ SYM equations

The harmonic superspaces for the $N = 4$ SYM-theory have been discussed in Refs.[8, 9, 10]. It has been shown that the G- and H-analytic Abelian on-shell superfield strength lives in the harmonic superspace with (4+4) Grassmann coordinates. We shall use the analogy with the HSS description of the $N = 3$ SYM-equations [14] and consider the gauge invariance and geometric structure of the superfield $N = 4$ equations. Stress that we do not know the off-shell superfield structure of the $N = 4$ SYM-theory in contrast to the $N = 3$ case [5], although these theories have the same physical component fields.

In order to study the light-cone SYM-conditions, one can use the non-covariant representation of the $D = 4,\ N = 4$ Grassmann coordinates $\theta_i^{\pm}$, $\bar{\theta}^{i\pm}$ where $i = 1,2,3,4$ are indices of the fundamental representations of $SU(4)$.

The $D = 4,\ N = 4$ SYM-constraints [6] have the following reduced-symmetry form:

$$\{\nabla_+^k, \nabla_+^l\} = 0 \ , \quad \{\bar{\nabla}_{k+}, \bar{\nabla}_{l+}\} = 0 \ , \quad \{\nabla_+^k, \bar{\nabla}_{l+}\} = 2i\delta_l^k \nabla_{\ddagger} \ , \tag{3.1}$$

$$\{\nabla_+^k, \nabla_-^l\} = W^{kl} \ , \quad \{\nabla_+^k, \bar{\nabla}_{l-}\} = 2i\delta_l^k \nabla_y \ , \tag{3.2}$$

$$\{\nabla_-^k, \bar{\nabla}_{l+}\} = 2i\delta_l^k \bar{\nabla}_y \ , \quad \{\bar{\nabla}_{k+}, \bar{\nabla}_{l-}\} = W_{kl} \ , \tag{3.3}$$

$$\{\nabla_-^k, \nabla_-^l\} = 0 \ , \quad \{\bar{\nabla}_{k-}, \bar{\nabla}_{l-}\} = 0 \ , \quad \{\nabla_-^k, \bar{\nabla}_{l-}\} = 2i\delta_l^k \nabla_{=} \ , \tag{3.4}$$

where ∇ are the covariant derivatives in the $(4|8,8)$-dimensional superspace, W_{kl} and W^{kl} are the gauge-covariant superfields constructed from the gauge connections. These superfields satisfy the subsidiary conditions

$$W^{ik} \equiv \overline{W_{ik}} = -\frac{1}{2}\varepsilon^{ikjl}W_{jl} \ . \tag{3.5}$$

The equations of motion for the superfield strengthes follow from the Bianchi identities

$$\begin{aligned}
&\nabla^i_\pm W^{kl} + \nabla^k_\pm W^{il} = 0 \ , \\
&\bar\nabla_{i\pm}W^{kl} = \frac{1}{2}(\delta^k_i\bar\nabla_{j\pm}W^{jl} - \delta^l_i\bar\nabla_{j\pm}W^{jk}) \ .
\end{aligned} \tag{3.6}$$

Let us consider the light-cone gauge conditions

$$A^k_+ = 0 \ , \quad \bar A_{k+} = 0 \ , \quad A_{\ddagger} = 0 \ . \tag{3.7}$$

We shall use the $SU(4)/U(1)^3$ harmonics [8, 10, 16] for the HSS interpretation of the non-Abelian $N=4$ constraints (3.1-3.4) by analogy with the Abelian case.

The $SU(4)/U(1)^3$ harmonics parametrize the corresponding coset space. They form an $SU(4)$ matrix and are defined modulo $U(1)\times U(1)\times U(1)$ transformations

$$u^1_i = u^{(1,0,1)}_i \ , \quad u^2_i = u^{(-1,0,1)}_i \ , \quad u^3_i = u^{(0,1,-1)}_i \ , \quad u^4_i = u^{(0,-1,-1)}_i \tag{3.8}$$

where i is the index of the quartet representation of $SU(4)$. The complex conjugated harmonics have opposite $U(1)$ charges

$$u^i_1 = u^{i(-1,0,-1)} \ , \quad u^i_2 = u^{i(1,0,-1)} \ , \quad u^i_3 = u^{i(0,-1,1)} \ , \quad u^i_4 = u^{i(0,1,1)} \ . \tag{3.9}$$

Note that we use indices $I, J = 1,2,3,4$ for the projected components of the harmonic matrix which do not transform with respect to the 'ordinary' $SU(4)$ transformations. The authors of Ref.[9] prefer to use the $SU(4)/S(U(2)\times U(2))$ harmonics for the $N=4$ theory.

The corresponding harmonic derivatives ∂^I_J act on these harmonics and satisfy the $SU(4)$ algebra.

The special conjugation of the $SU(4)$ harmonics has the following form:

$$u^1_i \leftrightarrow u^i_4 \ , \quad u^2_i \leftrightarrow u^i_3 \ , \quad u^3_i \leftrightarrow u^i_2 \ , \quad u^4_i \leftrightarrow u^i_1 \tag{3.10}$$

and the conjugation of the harmonic derivatives is

$$\partial^1_2 f \leftrightarrow -\partial^3_4\widetilde{f}, \quad \partial^1_4 f \leftrightarrow -\partial^1_4\widetilde{f}, \tag{3.11}$$

where $f(u)$ is an arbitrary harmonic function.

The analytic coordinates in the $N=4$ superspace $H(4,12|6,6)$ are

$$\begin{aligned}
&\zeta = (X^{\ddagger},\ X^{=},\ Y,\ \bar Y\ |\theta^\pm_2,\ \theta^\pm_3,\ \theta^\pm_4,\ \bar\theta^{1\pm},\ \bar\theta^{2\pm},\ \bar\theta^{3\pm}) \ , \quad X^{\ddagger} = x^{\ddagger} + i(\theta^+_4\bar\theta^{4+} - \theta^+_1\bar\theta^{1+}) \ , \\
&X^{=} = x^{=} + i(\theta^-_4\bar\theta^{4-} - \theta^-_1\bar\theta^{1-}) \ , \quad Y = y + i(\theta^+_4\bar\theta^{4-} - \theta^+_1\bar\theta^{1-}) \ , \\
&\bar Y = \bar y + i(\theta^-_4\bar\theta^{4+} - \theta^-_1\bar\theta^{1+}) \ , \quad \theta^\pm_I = \theta^\pm_k u^k_I \ , \quad \bar\theta^{I\pm} = \bar\theta^{\pm k}u^I_k \ .
\end{aligned} \tag{3.12}$$

The spinor derivatives have the following simple form in these coordinates:

$$D^1_\pm = \partial^1_\pm \ , \qquad \bar{D}_{4\pm} = \bar{\partial}_{4\pm} \ , \tag{3.13}$$

$$D^2_+ = \partial^2_+ + i\bar{\theta}^{2+}\partial_{\ddagger} + i\bar{\theta}^{2-}\partial_{Y} \ , \quad D^2_- = \partial^2_- + i\bar{\theta}^{2+}\bar{\partial}_{Y} + i\bar{\theta}^{2-}\partial_{=} \ , \tag{3.14}$$

$$\bar{D}_{1+} = \bar{\partial}_{1+} + 2i\theta^+_1\partial_{\ddagger} + 2i\theta^-_1\bar{\partial}_{Y} \ , \quad \bar{D}_{1-} = \bar{\partial}_{1-} + 2i\theta^+_1\partial_{Y} + 2i\theta^-_1\partial_{=} \ . \tag{3.15}$$

The corresponding harmonic derivatives are

$$D^1_2 = \partial^1_2 + i\theta^+_2\bar{\theta}^{1+}\partial_{\ddagger} + i\theta^+_2\bar{\theta}^{1-}\partial_{Y} + i\theta^-_2\bar{\theta}^{1+}\bar{\partial}_{Y} + i\theta^-_2\bar{\theta}^{1-}\partial_{=}$$
$$-\theta^+_2\partial^1_+ - \theta^-_2\partial^1_- + \bar{\theta}^{1+}\bar{\partial}_{2+} + \bar{\theta}^{1-}\bar{\partial}_{2-} \ , \tag{3.16}$$

$$D^3_4 = \partial^3_4 + i\theta^+_4\bar{\theta}^{3+}\partial_{\ddagger} + i\theta^+_4\bar{\theta}^{3-}\partial_{Y} + i\theta^-_4\bar{\theta}^{3+}\bar{\partial}_{Y} + i\theta^-_4\bar{\theta}^{3-}\partial_{=}$$
$$-\theta^+_4\partial^3_+ - \theta^-_4\partial^3_- + \bar{\theta}^{3+}\bar{\partial}_{4+} + \bar{\theta}^{3-}\bar{\partial}_{4-} \ . \tag{3.17}$$

Other projections of the Grassmann and harmonic derivatives can be constructed analogously.

Let us consider the harmonic projections of the CB covariant derivatives and the corresponding connections

$$\nabla^I_+ = u^I_k\nabla^k_+ = D^I_+ \ , \qquad \bar{\nabla}_{I+} = u^j_I\bar{\nabla}_{j+} = \bar{D}_{I+} \ , \tag{3.18}$$

$$\nabla^I_- = u^I_k\nabla^k_- = D^I_- + \mathcal{A}^I_- \ , \qquad \bar{\nabla}_{I-} = u^j_I\bar{\nabla}_{j-} = \bar{D}_{I-} + \bar{\mathcal{A}}_{I-} \ . \tag{3.19}$$

Taking into account these relations we can transform the CB-constraints (3.1-3.4) to the equivalent (2,2)-dimensional set of the G-integrability relations:

$$\{\nabla^1_\pm, \nabla^1_\pm\} = \{\nabla^1_\pm, \bar{\nabla}_{4\pm}\} = \{\bar{\nabla}_{4\pm}, \bar{\nabla}_{4\pm}\} = 0 \ . \tag{3.20}$$

Thus, the $N=4$ SYM-geometry preserves the Grassmann (6,6) analyticity. It can be shown that the covariant (4,4)-analyticity of superfield strength $u^1_k u^2_k W^{ik}$ follows from the basic (6,6)-analyticity in the HSS geometric formalism.

Now we shall discuss the solution of the G-integrability relations

$$\mathcal{A}^1_\pm(v) = e^{-v}\left(D^1_\pm e^v\right) \ , \quad \bar{\mathcal{A}}_{4\pm}(v) = e^{-v}\left(\bar{D}_{4\pm}e^v\right) \ , \tag{3.21}$$

where $v(z,u)$ is the superfield bridge matrix.

The gauge transformations of the bridge

$$e^v \ \Rightarrow \ e^\lambda e^v e^{-\tau} \ , \tag{3.22}$$

contain the (6,6)-analytic AB-gauge parameters λ

$$(D^1_\pm, \bar{D}_{4\pm})\lambda = 0 \tag{3.23}$$

and the harmonic-independent constrained CB-gauge parameters τ.

Matrix e^v determines a transform of the CB-gauge superfields to the analytic basis (AB). The analytic gauge group acts on the harmonic connections in AB

$$\nabla^I_K = e^v D^I_K e^{-v} = D^I_K + V^I_K(v) \ , \tag{3.24}$$

$$\delta V^I_K = D^I_K\lambda + [V^I_K, \lambda] \ . \tag{3.25}$$

Our gauge choice $\mathcal{A}^1_+ = \bar{\mathcal{A}}_{4+} = 0$ corresponds to the following partial gauge conditions for the bridge:

$$(D^1_+, \bar{D}_{4+})v = 0 \ . \tag{3.26}$$

We treat bridge v as the basic on-shell superfield, so the SYM-equations of this approach are formulated for this superfield

$$[D^I_K, e^{-v}D^1_- e^v] = [D^I_K, e^{-v}\bar{D}_{4-}e^v] = 0 \ , \quad I < K. \tag{3.27}$$

The subsidiary condition (3.5) is equivalent to the reality condition for the harmonic projection of the superfield strength $u^1_i u^2_k W^{ik}$ [8, 9] and corresponds to the following equation in the bridge representation:

$$-D^2_+(e^{-v}D^1_- e^v) = \bar{D}_{3+}(e^{-v}\bar{D}_{4-}e^v) \ . \tag{3.28}$$

By analogy with the $N = 3$ formalism [14] one can choose the following light-cone gauge for the $N = 4$ bridge:

$$v = \theta^-_1 b^1 + \bar{\theta}^{4-}\bar{b}_4 + \theta^-_1\bar{\theta}^{4-}d^1_4 \ , \tag{3.29}$$

where the fermionic matrices $b^1, \bar{b}_4$ and the bosonic matrix d^1_4 are the (6,6) analytic superfields. This bridge is nilpotent

$$v^2 = \theta^-_1\bar{\theta}^{4-}[\bar{b}_4, b^1] \ , \quad v^3 = 0 \ , \tag{3.30}$$

$$e^{-v} = I - v + \frac{1}{2}v^2 = I - \theta^-_1 b^1 - \bar{\theta}^{4-}\bar{b}_4 + \theta^-_1\bar{\theta}^{4-}(\frac{1}{2}[\bar{b}_4, b^1] - d^1_4) \ . \tag{3.31}$$

In the gauge group $SU(n)$, our superfields satisfy the conditions

$$(b^1)^\dagger = \bar{b}_4 \ , \quad (d^1_4)^\dagger = -d^1_4 \ , \quad \mathrm{Tr}\ b^1 = \mathrm{Tr}\ d^1_4 = 0 \ . \tag{3.32}$$

Consider the parametrization of the basic spinor connections in our gauge

$$\mathcal{A}^1_-(v) = b^1 - \theta^-_1(b^1)^2 + \bar{\theta}^{4-}f^1_4 + \theta^-_1\bar{\theta}^{4-}[b^1, f^1_4] \ , \tag{3.33}$$

$$\bar{\mathcal{A}}_{4-}(v) = \bar{b}_4 - \bar{\theta}^{4-}(\bar{b}_4)^2 + \theta^-_1\bar{f}^1_4 - \theta^-_1\bar{\theta}^{4-}[\bar{b}_4, \bar{f}^1_4] \ , \tag{3.34}$$

where the following auxiliary superfields are introduced:

$$f^1_4 = d^1_4 - \frac{1}{2}\{b^1, \bar{b}_4\} \ , \quad \bar{f}^1_4 = -d^1_4 - \frac{1}{2}\{b^1, \bar{b}_4\} \ . \tag{3.35}$$

The H-analyticity equations $(D^1_2, D^1_3, D^2_4, D^3_4)\mathcal{A}^1_-(v) = 0$ are equivalent to the following (6,6)-analytic equations:

$$(D^1_2, D^1_3)b^1 = -(\theta^-_2, \theta^-_3)(b^1)^2 \ , \quad (D^2_4, D^3_4)b^1 = -(\bar{\theta}^{2-}, \bar{\theta}^{3-})f^1_4 \ , \tag{3.36}$$

$$(D^1_2, D^1_3)f^1_4 = (\theta^-_2, \theta^-_3)[f^1_4, b^1] \ , \quad (D^2_4, D^3_4)f^1_4 = 0 \ . \tag{3.37}$$

We shall discuss below the relations between the matrices b^1 and $\bar{b}_4$ which arise from the transform of the CB-condition (3.28) to the analytic representation.

Remember that the following covariant Grassmann derivatives are flat in the AB-representation of the gauge group before the gauge fixing:

$$e^{v}\nabla^{1}_{\pm}e^{-v}=D^{1}_{\pm}\;,\quad e^{v}\bar{\nabla}_{4\pm}e^{-v}=\bar{D}_{4\pm} \tag{3.38}$$

The harmonic connections in the bridge representations $V^{I}_{K}(v)$ (3.24) satisfy automatically the harmonic zero-curvature equations

$$D^{I}_{K}V^{J}_{L}-D^{J}_{L}V^{I}_{K}+[V^{I}_{K},V^{J}_{L}]=\delta^{J}_{K}V^{I}_{L}-\delta^{I}_{L}V^{J}_{K}\;. \tag{3.39}$$

Basic SYM-equations (3.27)are equivalent to the dynamical G-analyticity conditions

$$(D^{1}_{-},\bar{D}_{4-})V^{I}_{K}(v)=0\;,\quad I<K\;. \tag{3.40}$$

In gauge (3.29), these equations give us the following relations:

$$V^{1}_{2}(v)=\theta^{-}_{2}b^{1}\;,\quad V^{1}_{3}(v)=\theta^{-}_{3}b^{1}\;, \tag{3.41}$$

$$V^{3}_{4}=(V^{1}_{2})^{\dagger}=-\bar{\theta}^{3-}\bar{b}_{4}\;,\quad V^{2}_{4}=(V^{1}_{2})^{\dagger}=-\bar{\theta}^{2-}\bar{b}_{4}\;, \tag{3.42}$$

where all connections are nilpotent. Similar relations have been considered in the harmonic formalism of the $N=3$ SYM-theory [14].

One can also construct the non-analytic harmonic connections

$$e^{v}D^{2}_{1}e^{-v}=V^{2}_{1}=-\theta^{-}_{1}D^{2}_{1}b^{1}\;. \tag{3.43}$$

The conjugated harmonic connection depend, respectively, on matrix $\bar{b}_4$ only

$$V^{4}_{3}=(V^{2}_{1})^{\dagger}=-\bar{\theta}^{4-}D^{4}_{3}\bar{b}_{4}\;. \tag{3.44}$$

It is not difficult to show that the harmonic AB-connections V^{2}_{1} satisfies the partial (8,6)-analyticity condition

$$\bar{D}_{4\pm}V^{2}_{1}=0 \tag{3.45}$$

and the conjugated connection possesses the (6,8)-analyticity

$$D^{1}_{\pm}V^{4}_{3}=0\;. \tag{3.46}$$

The basic AB-superfield strengthes can be constructed in terms of the harmonic connections

$$w^{12}=-D^{1}_{+}D^{1}_{-}V^{2}_{1}=-D^{2}_{+}b^{1}\;, \tag{3.47}$$

$$w_{34}=-\bar{D}_{4+}\bar{D}_{4-}V^{4}_{3}=-\bar{D}_{3+}c_{4}\;. \tag{3.48}$$

They satisfy the non-Abelian G- and H-analyticity conditions which generalize the shortness conditions for the corresponding Abelian superfields [10].

The reality condition

$$w^{12}=-w_{34} \tag{3.49}$$

is equivalent to the single linear differential relation between the matrices b^1 and $\bar{b}_4$ which can be easily solved via the following representation with the anti-Hermitian (6,6)-analytic bosonic matrix A^{13}

$$b^1 = \bar{D}_{3+}A^{13} \;, \qquad \bar{b}_4 = D^2_+ A^{13} \;, \tag{3.50}$$

$$w^{12} \equiv -w_{34} = -D^2_+ \bar{D}_{3+} A^{13} \;. \tag{3.51}$$

Consider the evident relation

$$(b^1)^2 = \frac{1}{2}\bar{D}_{3+}[A^{13}, \bar{D}_{3+}A^{13}] \;. \tag{3.52}$$

Equations (3.36) generate the following relations for A^{13}

$$(D^1_2, D^1_3)A^{13} = \frac{1}{2}(\theta^-_2, \theta^-_3)[A^{13}, \bar{D}_{3+}A^{13}] \;. \tag{3.53}$$

Thus, the harmonic-superspace representation and light-cone gauge conditions simplify significantly the analysis of the $N = 4$ SYM-equations. We hope that this representation allows us to construct the interesting solutions of these equations.

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